Abstract: The problem of designing a robust three-axis missile autopilot that operates under aerodynamic roll angle uncertainty is addressed in this article. A finite number of local state-space models over an aerodynamic roll angle envelope are developed as a multi-model to represent uncertainty bounds. Two design methods with multi-objective output-feedback control are proposed. In the first approach, a classical three-loop autopilot structure is slightly modified for the multivariable autopilot design. The optimal gains in the autopilot structure are automatically obtained by using a co-evolutionary optimization method that addresses competing specifications and constraints. In the second approach, the mixed $H_2/H_\infty$ performance criteria are guaranteed by multi-objective control synthesis via optimization techniques. Both design approaches are used in non-linear simulations with variations in the aerodynamic roll angle to provide satisfactory performance as a three-axis missile autopilot.

Keywords: robust control, three-axis missile autopilot, multi-model approach

1 INTRODUCTION

The increased demand for highly manoeuvrable modern missiles requires an excellent autopilot mechanism over a large flight envelope. This requires the control objectives to be expressed in terms of the time and frequency domains that should be achieved in more severe environments, such as those with undesirable induced roll moments. This increased complexity, in addition to the expanded capabilities of a high-performance missile, has been considered in the autopilot design problem [1, 2]. In classical approaches, which separately design for the roll, pitch, and yaw axis and ignore cross-coupling effects, the influence of high-angle-of-attack aerodynamic phenomena on stability characteristics is significant. For instance, the performance of separate controllers has been limited in some flight boundaries, wherein the interaction between each of the channels was minimal.

Over the course of the past decade, there have been many developments in the field of missile autopilot design, ranging from classical to modern control approaches. Both the approximate feedback linearization and the asymptotic output tracking methods have been investigated and compared to classical regulators [3]. Various robust design methods have been proposed using $H_\infty$ control or linear parameter-varying control until recently [4–8]. An adaptive control design technique has been applied in combination with non-linear control [9], neural networks control [10, 11], and sliding mode control approaches [12]. In addition, evolutionary algorithms have also been used to automatically obtain gains for specific autopilot configurations [13, 14]. Although these approaches demonstrate improvements that involve a fraction of the flight envelope and provide useful
ways to address major design problems, a single controller will not satisfy both the performance and robustness requirements for an agile missile that has rapidly changing non-linear dynamics.

On the other hand, studies have been performed to analyse the non-linearities and cross-coupling with the controller design in a coupled manner. These studies have attempted to address the robustness problem using $H_\infty$ control technique [15–17], a two-cascaded structure with an integral action [18], and feedback linearization technique that is based on two-time scale separation [19]. These multivariable robust control techniques can cover a larger flight envelope and offer a reasonable treatment of the fully coupled dynamics. However, agile missile autopilot designs have been achieved based on the assumption that the aerodynamic roll angle can be precisely estimated. From a practical point of view, because the angle of attack and the sideslip angle are not available from direct measurements, it is not trivial to design a highly precise state observer that is subject to three-axis accelerations or agile manoeuvres. For such cases, due to the effects of cross-coupling and the necessity output differentiation, some assumptions that have been made in the observer design are not always feasible.

To address these problems, two different three-axis autopilot design techniques are proposed for the development of an agile missile controller. The first design is based on the classical output-feedback control approach. A conventional three-loop autopilot configuration is slightly modified with a cross-feed loop so as to compensate for the cross-coupling effects that are induced from the variations in the aerodynamic roll angle. The optimal gains are automatically obtained by using a co-evolutionary optimization that can manage competing specifications and constraints. The other design uses a mixed $H_2/H_\infty$ output-feedback control approach. Because high-angle-of-attack aerodynamics are highly non-linear and very imprecise, a mix of the $H_2$ and $H_\infty$ criteria is selected to guarantee the performance of the three-axis missile autopilot. An upper bound of the suboptimal $H_2$ and $H_\infty$ performance is obtained by linear matrix inequality (LMI) conditions. In both autopilot design methods, a multi-model approach is used to avoid the need for the estimation of the aerodynamic roll angle. The contribution of the paper can be summarized as follows.

1. The three-axis missile autopilot design using evolutionary and LMI optimization techniques, which enhances robust performance against cross-coupling effects over a flight envelope, has been developed.

2. To account for model uncertainties, a finite set of possible linear time invariant (LTI) models at fixed operating conditions is considered instead of a single nominal LTI model. This formulates a problem of robust control design instead of a problem of estimating the aerodynamic roll angle. The application of the multi-model approach to three-axis missile autopilot design is almost unexplored in the literature.

3. A qualitative comparison between two autopilot design approaches is described as guidelines for evaluating the potential of each design approach in terms of its performance, robustness, and complexity.

Conventional approaches in modern robust control are not adequate for the problems associated with parameter variations due to changes in flight operating conditions, since the parameter variations caused by cross-coupling effects cannot be modelled properly. The proposed multi-model approaches are well suited for modern robust control autopilot designs in that they reduce the burden of the designer in representing the aerodynamic uncertainties [20]. Furthermore, the performance of the autopilot can be measured in terms of the worst case or average of these models’ responses. Both the classical and mixed $H_2/H_\infty$ output-feedback control techniques in this article are based on multi-objective control design concepts that satisfy the desired specifications in the time and frequency domains while accounting for the aerodynamic roll angle uncertainty. The underlying optimization techniques based on evolutionary algorithms and LMIs have been applied to the control system design effectively [14, 21, 22].

This article is organized as follows. In Section 2, the formulation of multi-model approaches for the missile control problem is presented. Section 3 describes the classical output-feedback controller design based on co-evolutionary optimization. Section 4 involves a mixed $H_2/H_\infty$ output-feedback controller design using LMI-based multi-objective control techniques. Section 5 presents the controller optimization results and its performance using high-fidelity non-linear simulations. The final section summarizes the overall discussion.

2 MISSILE CONTROL PROBLEM FORMULATION

2.1 Missile model

The missile model considered hereafter is a three-axis, non-linear model of a skid-to-turn cruciform missile with a high-angle-of-attack capability. It is assumed that the missile plant has rigid body
dynamics and six-degree-of-freedom with the fixed altitude and longitudinal velocity. This is reasonable for solid propellant missiles that perform gliding manoeuvres because the longitudinal velocity changes slowly after the combustion period [16]. The six-degree-of-freedom rigid-body equations of motions are expressed by the differential equations describing the translational and rotational motions as follows

\[
\begin{align*}
\dot{u} &= rv - qw + F_x/m \\
\dot{v} &= -ru + pw + F_y/m \\
\dot{w} &= qu - pv + F_z/m \\
\dot{p} &= M_x/I_{xx} \\
\dot{q} &= \left( I_{zz} - I_{xx} \right) pr + M_y/I_{yy} \\
\dot{r} &= \left( I_{yy} - I_{zz} \right) pq + M_z/I_{zz}
\end{align*}
\]

where \( u, v, \) and \( w \) are the longitudinal, lateral, and vertical body velocities, respectively; \( p, q, \) and \( r \) are the roll, pitch, and yaw body rates, respectively. The aerodynamic forces and moments about the body axes are represented by

\[
\begin{align*}
F_x &= -QS(C_x + C_{ax}) + ma_r \\
F_y &= QC_y \\
F_z &= QC_z \\
M_x &= QSD \left( C_i + \left( D/2V_m \right) C_p r \right) \\
M_y &= QSD \left( C_m + \left( D/2V_m \right) C_m q \right) \\
M_z &= QSD \left( C_n + \left( D/2V_m \right) C_n r \right)
\end{align*}
\]

The aerodynamic coefficients, \( C_p, C_{ax}, C_{bx}, C_y, C_z, C_{m}, C_{n}, C_{n}, \) and \( C_{n} \), are non-linear functions of Mach number \( M \), angle of attack \( \alpha \), sideslip angle \( \beta \), and fin deflections. The actuators are modelled by second-order transfer functions between the commanded fin deflection \( \delta_c \) and the actual fin deflection \( \delta \) as

\[
\frac{\delta}{\delta_c} = \frac{\omega_{act}^2}{s^2 + 2\zeta_{act}\omega_{act}s + \omega_{act}^2}
\]

with a natural frequency \( \omega_{act} \) of 30 Hz and a damping coefficient \( \zeta_{act} \) of 0.7.

Then, the state-space form of the missile model is written as

\[
\begin{align*}
\dot{x}_m &= A_m x_m + B_m u_m \\
y_m &= C_m x_m + D_m u_m
\end{align*}
\]

where

\[
\begin{align*}
x_m &= \left\{ \alpha, q, \beta, r, p, \phi, \delta_z, \delta_y, \delta_y, \delta_r \right\}^T \\
y_m &= \left\{ \eta_z, \eta_q, \eta_y, r, \phi, p \right\}^T \\
u_m &= \left\{ \delta_{xz}, \delta_{yz}, \delta_{xc} \right\}^T
\end{align*}
\]

In this expression, \( \delta_z, \delta_y, \) and \( \delta_r \) are the actual tail deflections in pitch, yaw, and roll axes, respectively. The inputs to the missile plant are commanded tail deflections \( \delta_{xz}, \delta_{yz}, \) and \( \delta_{xc} \). The system outputs are the pitch acceleration \( \eta_z, \) yaw acceleration \( \eta_y, \) bank angle \( \phi, \) and body angular rates. For the given missile problem, the total incidence angle \( \alpha' = \arccos(\cos \alpha \cos \beta) \) can be available as a scheduling parameter, but the aerodynamic roll angle \( \phi' = \arctan(\tan \beta / \sin \alpha) \) is not available through estimation techniques. The definitions for \( \alpha, \beta, \alpha', \) and \( \phi' \) are depicted in Fig. 1 with the aeroballistic wind coordinate [23].

The missile under consideration has a large flight envelope that includes high-angle-of-attack conditions. The asymmetric airflow on the control surfaces generates induced roll moments that result in control difficulties while operating in the

![Fig. 1 Aeroballistic wind coordinate systems](image-url)
high-angle-of-attack regime. It is difficult to estimate induced roll moments that are significantly affected by \( M, \phi, \text{ and } \alpha \), and fin configurations. The effects of this phenomenon can become more complicated as the missile manoeuvres in a low Mach number and high-angle-of-attack region. The angle of attack boundaries, which are mainly concerned in this article, include vortex-free and symmetric vortex flow as the angle of attack is increased from \( 0^\circ \) to \( 25^\circ \). The details of the cross-coupling effects that are caused by simultaneous pitch and yaw manoeuvring can be found in the authors’ previous work [16].

### 2.2 Multi-model approach

For a given missile control problem, the rapid estimation of the aerodynamic roll angle \( \phi \) with sufficient accuracy is not straightforward, whereas the total incidence angle \( \alpha' \) is estimated by an observer. Some difficulty arises because of the highly nonlinear aerodynamics in a high-angle-of-attack and a lack of appropriate sensors for both the angle of attack and the sideslip angle. Under this condition, a gain scheduling technique, which is commonly used in missile autopilot design to capture coupling effects, is not applicable. Instead, robust control approaches should be considered if the aerodynamic roll angle cannot be chosen as a scheduling parameter. As a result, the autopilot that was designed (with respect to the nominal model that was linearized at a specific aerodynamic roll angle) is expected to guarantee not only its robust stability but also the performance for the different aerodynamic roll angle trim conditions. To address this problem, a multi-model that was linearized at multiple flight conditions was constructed to address the aerodynamic roll angle uncertainty. If can be identified one autopilot that satisfies the stability and performance requirements for the multi-model plant, that autopilot can be considered to provide robust stability and performance under the aerodynamic roll angle uncertainty. This consideration converts the autopilot design problem into a multi-objective controller design problem. Some uncertainty that was due to other modelling errors can also be dealt with in the multi-model formulation. Furthermore, the classical and mixed \( H_2/\infty \) output-feedback control design methods are explored. In this article, a finite set of linearized models are considered as a state-space form of the multi-model plant

\[
[A_m, B_m, C_m, D_m] \in [A_i, B_i, C_i, D_i : i \in S]
\]

where \( S = \{1, \ldots, k\} \). Such a multi-model may result from the frequency bounds on the possible plant perturbations. As a consequence, it is desirable to find an output-feedback controller that satisfies the given performance requirements \( \text{via} \) the classical and mixed \( H_2/\infty \) control approaches. To accomplish this, the controller design procedure that is outlined in Fig. 2 is used and this design is discussed in detail in the following sections. It is expected that a missile autopilot that is designed in this manner has a very high probability of success in an actual plant due to its relatively small computational load.

### 2.3 Autopilot performance requirements

The goal of the autopilot design is to steer the missile to track the acceleration guidance commands that are generated by an outer loop and stabilize the missile airframe at a given bank angle. In particular, the stabilization of the bank angle is a critical requirement for controlling a highly manoeuvrable missile. The performance goals for the three-axis autopilot design are as follows.

1. Maintain robust stability over the operating range that is specified by \( \phi' < 180^\circ \). This robustness refers to the uncertainty in the induced roll moment, which has a significant effect in high-angle-of-attack regime.
2. Track the step acceleration commands with a time constant of less than 0.50 s, maximum overshoot and undershoot of less than 20 per cent, and steady-state error of less than 5 per cent.

### 3 CLASSICAL OUTPUT-FEEDBACK CONTROL SYNTHESIS

In this section, a multi-model approach is presented to classical output-feedback control synthesis using the same three steps that are outlined in Fig. 2. The first step involves developing an appropriate autopilot configuration to compensate for the effects of cross-coupling. Previously, missile autopilot design has been achieved using the roll, pitch, and yaw channels. However, highly coupled dynamics in agile maneuvers degrade the control performance of a decoupled autopilot structure. In this article, a three-loop autopilot with a cross-feed loop is proposed to address cross-coupling effects in the entire flight envelope. The second and third steps are related to obtaining a combination of gains that satisfy the control performance requirements using co-evolutionary optimization. This controller optimization method provides a useful way to address the multivariable feedback controller design problems.
3.1 Autopilot configuration

The proposed three-axis autopilot configuration with a cross-feed loop is shown in Fig. 3. For this autopilot configuration, the rate gyros feed the body rate measurements into the three-axis autopilot, whereas the accelerometers feed back the achieved acceleration commands. The ten autopilot gains must be chosen to satisfy the performance requirements: four pitch channel gains ($K_{P1}$, $K_{P2}$, $K_{P3}$, and $K_{P4}$), four yaw channel gains ($K_{Y1}$, $K_{Y2}$, $K_{Y3}$, and $K_{Y4}$), three roll channel gains, ($K_{R1}$, $K_{R2}$, and $K_{R3}$), and one cross-feed gain, $K_{ZYS}$. The expressions for the commanded deflections in the pitch $\delta_z$, yaw $\delta_y$, and roll channels $\delta_r$ can be written as

$$
\delta_z = \frac{(K_{P2}/s)(K_{P1}\eta_z - \eta_z) + (K_{P3}/s + K_{P4})q}{K_{Y2}(K_{Y1}\eta_y - \eta_y) + (K_{Y3}/s + K_{Y4})r}
$$

$$
\delta_y = \frac{K_{R1}(\phi_e - \phi) + K_{R2}p}{(1 + K_{R3}/s) + \delta_{zYS}}
$$

If the velocity of the missile is assumed to be constant, the induced roll moment is primarily a function of the total incidence and the aerodynamic roll angles. In particular, the induced roll moment can be described by a sinusoidal function of the aerodynamic roll angle and attains a maximum value if the aerodynamic roll angle is equal to $22.5^\circ$. Thus, the proposed three-axis autopilot with a classical output-feedback control is designed to compensate for the induced roll moment by predicting the aerodynamic roll angle from the acceleration commands. In this context, the form of the commanded deflection $\delta_{zYS}$ in a cross-feed loop is considered to be

$$
\delta_{zYS} = K_{ZYS} \sin(4\phi_p')
$$

where $\phi_p'$ is the aerodynamic roll angle predicted by $\phi_p' = \arctan(\eta_y/\eta_z)$.

3.2 Controller optimization

For classical output-feedback control synthesis, the gain selection problem is considered to be a parameter optimization problem of finding an optimal gain set. A conventional gain selection methodology [24] that is set by a linearized airframe for the selection of the system damping, time constant, and open-loop cross-over frequency is useful for a single-axis autopilot configuration; however, this methodology is not appropriate for coupled multivariable feedback control. Instead, the aforementioned 11 gains in the proposed autopilot configuration are automatically obtained by using the co-evolutionary augmented Lagrangian method (CEALM) [25] which can address...
complicated specifications and constraints. Because the CEALM can obtain the global minimum within a given search space and does not require gradient information or initial guesses for both states and co-states, it has been successfully applied to flight control system optimization problems [14, 27, 26].

Solutions to the optimization problem depend on the cost function and its constraints. For autopilot design problems that use CEALM, the solutions are selected by considering the desired performance and the system requirements. The three-axis autopilot design problem with the multi-model approach is formulated as a parameter optimization problem. Generally, as the number of objectives increases, trade-offs may become complex and difficult to quantify. Furthermore, there is a high dependence on the insight and preferences of the designer throughout the optimization cycle. In this article, to maximize the autopilot performance and address the set of models in a multi-model, equation (10) is defined as a cost function for the missile autopilot gain optimization based on numerical studies in the authors’ previous work [26]

\[
J = \sum_{i=1}^{k} I_i = \sum_{i=1}^{k} \left[ (w_i t_s + w_{p_o} P_o + w_{p_u} P_u)_{c} + (w_i t_s + w_{p_o} P_o + w_{p_u} P_u)_{c} ight] + \int_{t_0}^{t_f} \left[ w_{c_e} (\eta_{c_e} - \eta_c)^2 + w_{c_y} (\eta_{c_y} - \eta_y)^2 + w_{\phi} (\phi_c - \phi)^2 \right] dt 
\]

The cost function \( J \) is considered to minimize the errors between the commands and actual responses from the initial time to the final time. Other specifications that were incorporated into the design procedure with weighting coefficients are the settling time \( t_s \), maximum overshoot \( P_o \), and undershoot \( P_u \) in the time domain. This weighted sum method converts the multi-objective design problem for the multi-model into a scalar problem by constructing a weighted sum of all the objectives. The stability

![Diagram](image-url)
margins, which are given in terms of the gain and phase margins in the frequency domain, are considered as the constraints with the maximum overshoot and undershoot as follows

\[
\begin{align*}
GM & > GM_d \\
PM & > PM_d \\
P_o & < P_{od} \\
P_u & < P_{uid}
\end{align*}
\]  

(11)

4 MIXED \( H_2/H_{\infty} \) OUTPUT-FEEDBACK CONTROL SYNTHESIS

This section gives an overview of the multi-model approach to the mixed \( H_2/H_{\infty} \) output-feedback control synthesis, as outlined in Fig. 2. For this design approach, the cost function selection problem in classical output-feedback control synthesis becomes an LMI formulation problem and includes the \( H_{\infty} \) and \( H_2 \) performances, which are subject to robustness for the given multi-model description. The \( H_{\infty} \) performance is convenient for enforcing the robustness of the model uncertainty and expressing frequency-domain specifications. The \( H_2 \) performance is useful for addressing the output (or error) power of the generalized system due to a unit intensity white noise input. For the output-feedback case, the dimensions of the resulting controller that solve the \( H_2/H_{\infty} \) problem do not exceed the dimensions of the generalized plant.

To implement the proposed mixed \( H_2/H_{\infty} \) output-feedback control, one of the generalized plants for the multi-model composition of the following state-space form is represented

\[
\begin{bmatrix}
\dot{x} \\
z \\
y \\
u
\end{bmatrix} =
\begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}
\begin{bmatrix}
x \\
\omega \\
u
\end{bmatrix}
\]

(12)

where \( A \in \mathbb{R}^{n \times n} \), \( D_{12} \in \mathbb{R}^{m \times m} \), and \( D_{21} \in \mathbb{R}^{p \times m} \), while \( x, u, z, y, \) and \( w \) are the state, control input, controlled output, measurement output, and exogenous input, respectively. Then, the multi-objective control problem is to find an LTI control law \( u = K(s)y \) that minimizes an upper bound for the \( H_2 \) gain subject to the \( H_{\infty} \) gain constraint. In a mixed \( H_2/H_{\infty} \) control problem, the following assumptions [20] are made for each of the generalized plants.

1. \( (A, B_2, C_2) \) is stabilizable and detectable.
2. \( D_{12} \) and \( D_{21} \) have full rank.
3. \( A - j\omega I \\
\begin{bmatrix}
B_2 \\
C_1 & D_{12}
\end{bmatrix}
\] has full column rank for all \( \omega \).
4. \( A - j\omega I \\
\begin{bmatrix}
B_1 \\
C_2 & D_{21}
\end{bmatrix}
\] has full row rank for all \( \omega \).
5. \( D_{11} = 0 \) and \( D_{22} = 0 \).

Assumption (A1) is required for the existence of a stabilizing controller \( K \), and assumption (A2) is sufficient to guarantee the controller is proper and realizable. Assumptions (A3) and (A4) ensure that the controller does not attempt to cancel poles or zeros on the imaginary axis which would result in closed-loop instability. Assumption (A5) is typically made in \( H_2 \) control.

4.1 Multi-objective LMI formulation

LMI formulations for the multi-model approach to the mixed \( H_2/H_{\infty} \) output-feedback control are introduced. For brevity, multi-objective LMI formulations for a single-model case are described. After than extension of LMI formulations to a multi-model case is established subsequently. If all design objectives are formulated in terms of a common Lyapunov function, controller design is emerged to solve a system of LMIs. The resulting LMI formulations are induced from the multi-objective control synthesis [21, 22].

The LTI controller \( K(s) \) can be represented in state-space form by

\[
\begin{align*}
\dot{x}_K(t) &= A_K x_K(t) + B_K y(t) \\
u(t) &= C_K x_K(t) + D_K y(t)
\end{align*}
\]

(13)

For LMI approach to multi-objective synthesis, nonlinear terms added in the output feedback case should be eliminated by some appropriate change of controller variables. This change of controller variables is implicitly defined in terms of the Lyapunov matrix \( P \).

\[
P = \begin{bmatrix}
Y & N^T \\
N & V
\end{bmatrix}, \quad P^{-1} = \begin{bmatrix}
X & M^T \\
M & U
\end{bmatrix}
\]

(14)

where \( X \in \mathbb{R}^{n \times n} \) and \( Y \in \mathbb{R}^{m \times m} \) are symmetric. The new controller variables can be written by

\[
\begin{align*}
\dot{A}_K := & \quad N A_K M^T + NB_K C_2 X + Y B C_K M^T \\
& \quad + Y (A + B_2 D_K C_2) X, \\
\dot{B}_K := & \quad NB_K + Y B_2 D_K, \\
\dot{C}_K := & \quad C_K M^T + D_K C_2 X, \\
\dot{D}_K := & \quad D_K.
\end{align*}
\]

(15)

Then, the mixed \( H_2/H_{\infty} \) synthesis would be finding \( X, Y, \dot{A}_K, \dot{B}_K, \dot{C}_K, \) and \( \dot{D}_K \) such that equations (16) to (20) are hold, while minimizing \( \gamma \) and \( v \).

\[
\begin{bmatrix}
\psi_{11} & \psi_{21} & \psi_{31} & \psi_{41} \\
\psi_{22} & \psi_{22} & \psi_{32} & \psi_{42} \\
\psi_{31} & \psi_{32} & \psi_{33} & \psi_{34} \\
\psi_{41} & \psi_{42} & \psi_{43} & \psi_{44}
\end{bmatrix} < 0
\]

(16)
LMI problem of the form

\[
\begin{bmatrix}
\Gamma_{11} & \Gamma_{12}^T & \Gamma_{13}^T \\
\Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\
\Gamma_{31} & \Gamma_{32} & \Gamma_{33}
\end{bmatrix} < 0
\]  

(17)

\[
\begin{bmatrix}
X & I & * \\
I & Y & * \\
C_1X + D_{12}\hat{C}_K & C_1 + D_{12}\hat{D}_K C_2 & Q
\end{bmatrix} > 0
\]  

(18)

\[Tr(Q) < v\]  

(19)

\[(D_{11} + D_{12}\hat{D}_K D_{21}) = 0\]  

(20)

with the shorthand notation
\[
\psi_{11} := AX + XA^T + B\hat{C}_K + (B\hat{C}_K)^T
\]

\[
\psi_{21} := \hat{A}_K + (A + B_2\hat{D}_K C_2)^T
\]

\[
\psi_{22} := A^TY + YA + \hat{B}_K C_2 + (\hat{B}_K C_2)^T
\]

\[
\psi_{31} := (B_1 + B_2\hat{D}_K D_{21})^T
\]

\[
\psi_{32} := (YB_1 + \hat{B}_K D_{21})^T
\]

\[
\psi_{33} := -\gamma I
\]

\[
\psi_{41} := C_1X + D_{12}\hat{C}_K
\]

\[
\psi_{42} := C_1 + D_{12}\hat{D}_K C_2
\]

\[
\psi_{43} := D_{11} + D_{12}\hat{D}_K D_{21}
\]

\[
\psi_{44} := -\gamma I
\]  

(21)

\[
\Gamma_{11} := AX + XA^T + B_2\hat{C}_K + (B_2\hat{C}_K)^T
\]

\[
\Gamma_{21} := \hat{A}_K + (A + B_2\hat{D}_K C_2)^T
\]

\[
\Gamma_{22} := A^TY + YA + \hat{B}_K C_2 + (\hat{B}_K C_2)^T
\]

\[
\Gamma_{31} := (B_1 + B_2\hat{D}_K D_{21})^T
\]

\[
\Gamma_{32} := (YB_1 + \hat{B}_K D_{21})^T
\]

\[
\Gamma_{33} := -I
\]  

(22)

The previous formulations are directly extended to uncertain systems described by the multi-model approach. Note that LMI conditions for \(H_2\) and \(H_\infty\) performances over the set of linear models are obtained similarly by writing equations (16) to (20) for each of the linear model components.

### 4.2 Controller computation

The multi-objective output-feedback synthesis is an LMI problem of the form

Minimize \(\gamma + v\) over \(X, Y, \hat{A}_K, \hat{B}_K, \hat{C}_K, \hat{D}_K, \gamma, v\)

satisfying equations (16) to (20).

(23)

After solving the synthesis LMI formulations, the controller computation proceeds as follows:

1. Find non-singular matrix \(M, N\) to satisfy \(MN^T = I - XY\) via singular value decomposition.

2. Solve equation (15) for the controller \(K(s)\)

\[
\begin{aligned}
D_K &:= \hat{D}_K, \\
C_K &:= \left(\hat{C}_K - D_K C_1 X\right)M^{-T}, \\
B_K &:= N^{-1}\left(\hat{B}_K - YB_2 D_K\right), \\
A_K &:= N^{-1}\left(A_K - NB_K C_1 X - YB_2 C_1 M^T\right)
\end{aligned}
\]  

(24)

This LMI optimization problem can be efficiently solved using the LMI Control Toolbox [28]. The given LMI problem is solvable if and only if the LMI conditions given by equations (16) to (20) are feasible. Because \(H_\infty\) and \(H_2\) constraints are removed in the LMI problem, the LMI conditions become necessary and sufficient. Related proofs can be found in reference [21]. The solution of the LMI problem gives an upper estimate of the suboptimal \(H_\infty\) and \(H_2\) performance. For the output-feedback control synthesis, a set of controllers which depend on the system dynamics are obtained instead of a single controller, since equation (24) involves the terms of the system matrix \(A, B_2, C_2\). The dependency caused by the nature of output-feedback control cannot be removed by a change of variables directly. For this reason, controller switching between computed controllers is employed based on the predicted aerodynamic roll angle \(\phi_p^*_d\) from the acceleration commands. The points of the controller switching are determined by the selected operating points for the multi-model construction. This scheme can be considered as gain-scheduling control in that controller switching depends on commands, not state variables. Note that the predicted aerodynamic roll angle is also used in the cross-feed loop of the classical output-feedback controller.

### 4.3 Interconnected system model

The mixed \(H_2/H_\infty\) control approach for characterizing the closed-loop performance objectives is to measure the norms of the closed-loop transfer function matrices. Because a natural performance objective is to provide a closed-loop gain from exogenous influences, \(\omega\) (reference commands \(r\), sensor noise \(n\), and external force disturbances \(d\)), to the regulated variables (tracking errors \(e\) and control input \(u\)), the performance specifications for the closed-loop system in terms of multi-objective control can be defined as follows.

1. Guarantee an upper bound on the \(\gamma\) of the operator mapping \(z_\infty\) to \(\omega\).

2. Minimize an upper bound on the variance of \(z_\infty\) due to the disturbance \(\omega\).
These performance requirements of the closed-loop system are transformed into a multi-objective control framework with weighting functions. The weighting functions are selected so as to account for the relative magnitude of the signals, frequency dependence, and relative importance. This method allows the controller designer to apply classical loop-shaping concepts so as to obtain robust performance while optimizing the response near the system bandwidth to achieve robust stabilization. The weighted interconnection that incorporates the missile model $G(s)$ with the weighting functions, $W_{\text{ref}}$, $W_{\text{err}}$, $W_{\text{uo}}$ and $W_{\text{noise}}$ is depicted in Fig. 4. These frequency-dependent weights were chosen by a trial-and-error approach in order to maintain a sufficiently high bandwidth and minimize the steady-state error.

$W_{\text{err}}$ weights the difference between the response of the closed-loop system and the desired response. It is required that the tracking performance has accurate matching of the desired response at low frequency and less accurate matching at higher frequency. The weighting function $W_{\text{err}}$ is represented by

$$W_{\text{err}} = \left(\frac{s/\sqrt{M_{\text{err}} + \omega_{\text{err}}}}{s + \omega_{\text{err}}\sqrt{A_{\text{err}}}}\right)^n$$  \hspace{1cm} (25)

where $\omega_{\text{err}}$ is the minimum bandwidth frequency, $M_{\text{err}}$ the maximum peak magnitude, and $A_{\text{err}}$ the maximum steady-state tracking error. $W_{\text{ref}}$ represents desired responses for the closed-looped system and included in problem formulations with tracking requirements. For good command tracking response, this weight functions is a well-damped second-order system with specific natural frequency $\omega_{\text{ref}}$ and damping ratio $\zeta_{\text{ref}}$

$$W_{\text{ref}} = \frac{\omega_{\text{ref}}^2}{s^2 + 2\zeta_{\text{ref}}\omega_{\text{ref}}s + \omega_{\text{ref}}^2}$$  \hspace{1cm} (26)

$W_u$ is the respective size of the largest anticipated multiplicative plant perturbation. By introducing a multiplicative dynamic uncertainty model at the input to the missile plant, as shown in Fig. 4, robustness to unmodelled dynamics is covered. To prevent that the plant model with multiplicative uncertainty contains many plants that never occur, a trade-off between the selection of $W_u$ and multi-model composition. In this study, $W_u$ is chosen to be $W_u = \omega_{\Delta}(s)I_{3 \times 3}$, with a scalar-valued function $\omega_{\Delta}(s)$

$$\omega_{\Delta} = \frac{0.10s + 1}{0.005s + 1}$$  \hspace{1cm} (27)

The scalar valued function $\omega_{\Delta}$ represents a percentage error in the model and is small in magnitude at low frequency around 0.1 (10 per cent modelling error), and growing larger in magnitude at high frequency, over 2 (200 per cent modelling error).

5 NUMERICAL RESULTS

In this section, the proposed design strategies, from the classical to the mixed $H_2/H_\infty$ control, are applied to the three-axis missile autopilot design problem. The controller optimization results that were subjected to the design requirements are presented. The robustness of these controllers against aerodynamic roll angle uncertainty is tested. The non-linear simulations that account for cross-coupling effects verify the proposed controllers. A qualitative controller comparison is given for two different multi-model approaches based on the numerical results.

5.1 Controller optimization results

The set of operating conditions for which the controllers are designed corresponds to five values of $\phi'$,
which are given in Table 1 for the multi-model composition. These values were chosen by equidistant gridding over the variation of $\phi'$ from zero to $\pi/2$. These are reasonable values because the induced roll moment attains a maximum value if $\phi'$ is 22.5° and 67.5°. For any value of $\phi'$, $\alpha'$, and $V_M$, the corresponding equilibria and linearized state-space model can be computed in a typical fashion. For the multi-model that was linearized at each of these five operating conditions, the three-axis missile autopilot is designed to satisfy the performance requirements by the use of the two proposed autopilot design methods.

For the case of the classical output-feedback control design, the optimal gains for the modified three-loop autopilot configuration were obtained using the CEALM. The search range of each gain was $[-10, 10]$. The population sizes for the CEALM were selected as $\mu = 9$ (parents) and $\lambda = 40$ (offspring). The weighting coefficients selected empirically for the cost function are $w_r = 1$, $w_p = 1$, $w_n = 1000$, $w_n = 100$, and $w_s = 5000$. The values of the four constraints are determined by the performance goals as $GM_d = 6$ dB, $PM_d = 30^\circ$, $P_{\text{out}} = 20$ per cent, and $P_{\text{std}} = 20$ per cent. Table 2 depicts the best of the optimal gains from five runs. The controller specifications with these gains meet the autopilot requirements, as can be observed in Table 3.

For the case of the mixed $H_2/H_\infty$ output-feedback control design, the parameters of the weighting functions for the interconnected system model are chosen as $n = 2$, $\omega_{\text{ref}} = 50$, $M_{\text{ref}} = 10^{-1}$, $A_{\text{ref}} = 10^{-4}$, $\omega_{\text{ref}} = 25$, and $\zeta_{\text{ref}} = 0.707$. The weight functions of noise signals are $W_{\text{noise}}(s) = 10^{-4}$ and $W_{\text{noise}}(s) = 10^{-4}$. Figure 5 shows the frequency responses of these weighting functions. The optimized autopilot specifications, including values of the $H_2$ and $H_\infty$ norms, are listed in Table 4. These results indicate that the designed controller not only satisfies the autopilot design requirements but also exhibits good tracking performance due to small optimized $\gamma$ and $v$.

To emphasize the motivation of the proposed multi-model approaches, the robustness against the aerodynamic roll angle uncertainty should be studied. Next, linear simulations of the designed controllers were applied for a set of 13 linear models, which were chosen by equidistant gridding over a variation of $\phi'$ from zero to $\pi$, including the nominal models. Step commands in $\eta_{xc}$ and $\eta_{yc}$ which was maintained at 1 g from 0 to 1.5 s was examined with $\phi_c = 0^\circ$, and the simulation results are presented in Figs 6 and 7. The above simulation results demonstrate that both the multi-model approaches exhibit satisfactory tracking performance and robustness under the conditions of aerodynamic roll angle uncertainty.

---

### Table 1 Trim conditions for local linearization

<table>
<thead>
<tr>
<th>Model</th>
<th>Model A</th>
<th>Model B</th>
<th>Model C</th>
<th>Model D</th>
<th>Model E</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi'$ (degree)</td>
<td>0.0</td>
<td>22.5</td>
<td>45.0</td>
<td>67.5</td>
<td>90.0</td>
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<tr>
<td>$\alpha'$ (degree)</td>
<td>15.4</td>
<td>15.4</td>
<td>15.4</td>
<td>15.4</td>
<td>15.4</td>
</tr>
<tr>
<td>$V_M$ (m/s)</td>
<td>680.6</td>
<td>680.6</td>
<td>680.6</td>
<td>680.6</td>
<td>680.6</td>
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</table>

### Table 2 Optimal gains for classical output-feedback configuration

<table>
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<tr>
<th>Pitch channel</th>
<th>Yaw channel</th>
<th>Roll channel</th>
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<tr>
<td>$K_{\text{Pr}}$</td>
<td>$K_{\text{Pr}}$</td>
<td>$K_{\text{Pr}}$</td>
</tr>
<tr>
<td>0.1535</td>
<td>1.535</td>
<td>0.2779</td>
</tr>
<tr>
<td>2.6568</td>
<td>2.6568</td>
<td>2.6568</td>
</tr>
<tr>
<td>0.1133</td>
<td>-0.1133</td>
<td>2.5353</td>
</tr>
<tr>
<td>1.2750</td>
<td>1.2750</td>
<td>-0.0002</td>
</tr>
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</table>

### Table 3 Optimized autopilot specifications with classical output-feedback control

<table>
<thead>
<tr>
<th>Model</th>
<th>Model A</th>
<th>Model B</th>
<th>Model C</th>
<th>Model D</th>
<th>Model E</th>
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</thead>
<tbody>
<tr>
<td>Cost $J_f$</td>
<td>119.3</td>
<td>76.64</td>
<td>67.86</td>
<td>74.01</td>
<td>109.9</td>
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<tr>
<td>$GM_r$ (dB)</td>
<td>14.5</td>
<td>14.6</td>
<td>15.1</td>
<td>15.1</td>
<td>12.5</td>
</tr>
<tr>
<td>$PM_r$ (degree)</td>
<td>36.8</td>
<td>37.8</td>
<td>37.8</td>
<td>35.9</td>
<td>38.4</td>
</tr>
<tr>
<td>$GM_r$ (dB)</td>
<td>13.2</td>
<td>15.3</td>
<td>15.1</td>
<td>14.6</td>
<td>14.5</td>
</tr>
<tr>
<td>$PM_r$ (degree)</td>
<td>38.5</td>
<td>35.8</td>
<td>38.1</td>
<td>37.2</td>
<td>36.4</td>
</tr>
<tr>
<td>$GM_y$ (dB)</td>
<td>6.29</td>
<td>6.55</td>
<td>7.73</td>
<td>8.58</td>
<td>6.00</td>
</tr>
<tr>
<td>$PM_y$ (degree)</td>
<td>33.6</td>
<td>30.0</td>
<td>31.7</td>
<td>37.0</td>
<td>33.9</td>
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<tr>
<td>$(t)<em>{y</em>{zc}}$ (s)</td>
<td>0.136</td>
<td>0.127</td>
<td>0.128</td>
<td>0.298</td>
<td>0.458</td>
</tr>
<tr>
<td>$(t)<em>{y</em>{zc}}$ (s)</td>
<td>0.483</td>
<td>0.371</td>
<td>0.127</td>
<td>0.180</td>
<td>0.123</td>
</tr>
<tr>
<td>$(P_{\text{Pr}})$ (s)</td>
<td>0.514</td>
<td>4.85</td>
<td>1.91</td>
<td>2.23</td>
<td>4.53</td>
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<tr>
<td>$(P_{\text{Pr}})$ (s)</td>
<td>0.656</td>
<td>0.693</td>
<td>2.75</td>
<td>8.52</td>
<td>2.78</td>
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<tr>
<td>$(P_{\text{Pr}})$ (s)</td>
<td>10.3</td>
<td>9.99</td>
<td>9.59</td>
<td>10.1</td>
<td>13.0</td>
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<tr>
<td>$(P_{\text{Pr}})$ (s)</td>
<td>13.1</td>
<td>9.90</td>
<td>9.54</td>
<td>9.32</td>
<td>10.0</td>
</tr>
</tbody>
</table>

Notes: $J_f$ = 447.7, $GM_{\text{min}} = 6.00$ dB, $PM_{\text{min}} = 30.0^\circ$

$t_{\text{max}} = 0.483$ s, $P_{\text{r, max}} = 13.0\%$, and $P_{\text{r, max}} = 13.1\%$. 

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D-Y Won, M-J Tahk, Y-H Kim, and H J Kim

Proc. IMechE Vol. 000 Part G: J. Aerospace Engineering
5.2 Non-linear simulation results

The performance of the optimized autopilot with the proposed approaches was verified via six-degree-of-freedom flight simulations. The simulation program includes the non-linear time-varying dynamics of the missile. In order to examine the effects of cross-coupling under the conditions of aerodynamic roll angle uncertainty, a sequence of acceleration commands with variations in the aerodynamic roll angle from zero to $\pi/2$ was simulated. Figures 8 and 9 depict the responses of the classical and mixed $H_2/H_\infty$ output-feedback controllers. The bank angle responses reduce to less than $5^\circ/C_{14}$ and approach to $0^\circ/C_{14}$ at the end, while the pitch and yaw acceleration commands are tracked simultaneously. From the non-linear simulation results, the performance goals are satisfied for both the controller design methods under the circumstance that the aerodynamic roll angle is not directly available.

5.3 Qualitative controller comparison

A qualitative comparison was made for the proposed three-axis autopilot design approaches. This comparison is not intended to be a method for determining the better controller because this is generally not possible. Instead, their similarity is demonstrated in various aspects so as to explain if and how their performances differ. A comparison for each design approach is shown in Table 5 in terms of the optimization technique, controller order, robustness, and complexity.

From a practical point of view, implementation of the resulting controller via mixed $H_2/H_\infty$ output-feedback can be computationally expensive. Since the relatively high controller order is obtained from the given interconnected system model, the controller designed with robust control synthesis should be reduced to a lower order controller. In literature, there are various controller reduction methods which have been attempted to capture the $H_\infty$ performance criterion [29, 30]. A comparative study of several controller reduction methods in $H_\infty$ frameworks conducted by Zhou [29] provides a useful introduction to the issue.

The qualitative comparison shows that the design methods provide reasonable results, while each has own usefulness and limitations. Basically, the principal advantages of the proposed multi-model approaches to missile autopilot design are twofold: a reduction in the burden for the designer in representing non-linear and highly coupled aerodynamic uncertainties, and a guarantee of performance with uncertainties in terms of worst case or average responses. On the other hand, it implies that the burden is moved into computational complexities of optimization algorithms or controller implementation methods. Note that numerical results from this study indicate that the computational loads are not overly high, and they are managed effectively by current evolutionary and LMI optimization techniques.

6 CONCLUSION

In this article, output-feedback controllers were designed for a high-angle-of-attack missile autopilot if the angle of attack, sideslip angle, and aerodynamic

| Table 4 | Optimized autopilot specifications with mixed $H_2/H_\infty$ output-feedback control |
|---------|---------------------------------|-----------------|----------------|-----------------|-----------------|
| Model   | Model A                         | Model B         | Model C        | Model D         | Model E         |
| $t_{z\text{max}}$ (s) | 0.119                          | 0.191            | 0.197          | 0.132           | 0.394           |
| $t_{\text{u\text{max}}}$ (s) | 0.473                          | 0.410            | 0.193          | 0.191           | 0.127           |
| $\left|P_{\text{z\text{max}}}(\%\right| | 18.4                        | 11.6             | 11.4           | 15.5            | 10.6            |
| $\left|P_{\text{u\text{max}}}(\%\right| | 5.62                        | 6.90             | 7.29           | 6.54            | 5.27            |
| $\left|P_{\text{err\text{max}}}(\%\right| | 7.92                        | 14.1             | 13.5           | 9.87            | 18.6            |
| $\left|P_{\text{err\text{max}}}(\%\right| | 4.15                        | 6.00             | 7.20           | 7.73            | 7.31            |

Notes: $\gamma = 2.307$, $\nu = 0.996$.

$t_{z\text{max}} = 0.473$ s, $P_{\text{z\text{max}}} = 18.6\%$, and $P_{\text{u\text{max}}} = 7.73\%$. 

Three-axis missile autopilot design
roll angle measurements are not directly available. The multi-model approaches that were developed to address the aerodynamic roll angle uncertainty were motivated by a practical point of view. The proposed design procedure, which is based on classical and mixed $H_2/H_\infty$ control methods, is useful for handling multiple design criteria that are translated from missile autopilot requirements.

**Fig. 6** Step responses of the classical output-feedback control to operating point variation
These multi-model approaches can be applied to practical autopilot design problems without a heavy computational burden. The findings from the simulation results indicate that the designed autopilot provides satisfactory performances for aerodynamic roll angle estimation uncertainties and non-linearities over high-angle-of-attack flight boundaries. Although the simulation results presented in this

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**Fig. 7** Step responses of the mixed $H_2/H_\infty$ output-feedback control to operating point variation
article only illustrate the formulations for the compensation of cross-coupling effects under aerodynamic roll angle uncertainty, further testing is required to arrive at any conclusions regarding the effectiveness of multi-model approaches to missile autopilot design problems. A qualitative comparison between the proposed control methods provides guidelines to controller designers for evaluating the

**Fig. 8** Missile dynamic response of the classical output-feedback control to a sequence of acceleration commands with the aerodynamic roll angle variation from zero to $\pi/2$
Fig. 9  Missile dynamic response of the mixed $H_2/H_\infty$ output-feedback control to a sequence of acceleration commands with the aerodynamic roll angle variation from zero to $\pi/2$
Table 5 Comparison of two autopilot design approaches

<table>
<thead>
<tr>
<th>Design approach</th>
<th>Classical output-feedback</th>
<th>Mixed $H_\infty$ output-feedback</th>
</tr>
</thead>
<tbody>
<tr>
<td>Optimization technique</td>
<td>CEALM optimization (co-evolutionary algorithm)</td>
<td>LMI optimization (interior-point algorithm)</td>
</tr>
<tr>
<td>Controller order ($n$)</td>
<td>Depends on configurations</td>
<td>Depends on weighting functions</td>
</tr>
<tr>
<td>Robustness</td>
<td>Gain and phase margins</td>
<td>$H_\infty$ norm</td>
</tr>
<tr>
<td>Design technique</td>
<td>Classical feedback design know-how (difficult to guarantee robustness with respect to model uncertainties)</td>
<td>Multivariable feedback design methodology (difficult to tradeoff between time domain performances)</td>
</tr>
</tbody>
</table>

potential of each control method in terms of its performance, robustness, and complexity.

ACKNOWLEDGEMENT

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REFERENCES


**APPENDIX**

**Notation**

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<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$a_T$</td>
<td>thrust</td>
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<tr>
<td>$C_{x\alpha}$, $C_{y\beta}$, $C_{z\delta}$, $C_{m\phi}$, $C_{n\psi}$, $C_{r\phi}$</td>
<td>aerodynamic coefficients</td>
</tr>
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<td>$D$</td>
<td>reference diameter (m)</td>
</tr>
<tr>
<td>$F_x$, $F_y$, $F_z$</td>
<td>aerodynamic forces</td>
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<tr>
<td>$GM$</td>
<td>gain margin (dB)</td>
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<td>$I_{xx}$, $I_{yy}$, $I_{zz}$</td>
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<td>number of elements in a set of linear models</td>
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<td>missile mass (kg)</td>
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**Subscripts**

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<tr>
<td>$c$</td>
<td>commanded value of variable</td>
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<tr>
<td>$d$</td>
<td>desired value of variable</td>
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<td>$i$</td>
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